Question No. 1 of 10

Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #01	**1.** Evaluate the limit (if possible): $\lim_{x \to 0} \frac{\sin x - x}{x^3}$ (A) 0 (B) $\frac{1}{6}$ (C) $-\frac{1}{6}$ (D) −1 (E) 1
Feedback on Each Answer Choice	B. Incorrect! Remember you must check every time L'Hopital's Rule is used whether or not the new limit is an indeterminate form or not before using L'Hopital's Rule again.
	B. Incorrect! The derivative of cos x in the second step is −sin x not sin x so the sign will be negative 1/6 at the end.
	C. Correct! Good job, you needed to use L'Hopital's Rule three times to obtain this result.
	D. Incorrect! Don't forget to include the denominator and be careful or the signs. Cos 0 = 1, so 1 −cos 0 = −1.
	E. Incorrect! Don't forget to include the denominator which is 6 when the final application of L'Hopital's Rule is applied.
Solution	To evaluate this limit we first identify the limit form. We note that as $x \to 0$, $x^3 \to 0$ and $\sin x - x \to 0$ so we identify this as a 0/0 indeterminate form. Consequently, we can apply L'Hopital's Rule. Hence, $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x - x)}{\frac{d}{dx}(x^3)}$ $= \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$ Once again, we notice that as $x \to 0$, $3x^2 \to 0$ and $\cos x - 1 \to 0$ so this is still a 0/0 indeterminate form. We can apply L'Hopital's Rule again to produce $= \lim_{x \to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(3x^2)} = \lim_{x \to 0} \frac{-\sin x}{6x}$ Once again, we notice that as $x \to 0$, $6x \to 0$ and $-\sin x \to 0$ so this is still a 0/0 indeterminate form. We can apply L'Hopital's Rule again to produce $= \lim_{x \to 0} \frac{\frac{d}{dx}(-\sin x)}{\frac{d}{dx}(6x)} = \lim_{x \to 0} \frac{-\cos x}{6}$ Taking the limit yields $= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{\cos 0}{6} = -\frac{1}{6}$ **Solution** $\lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$

Instruction: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.

Question #02	**2.** Find the limit: $\lim_{x \to \pi/2} \left[\tan x \cdot \ln(\sin x)\right]$ (A) −1 (B) 0 (C) 1 (D) $-\cos x \sin x$ (E) The limit doesn't exist
Feedback on Each Answer Choice	A. Incorrect! Don't forget to use the chain rule when differentiating ln(sin *x*). The derivative should be (1/sin x) · cos *x*.
	B. Correct! Nice work! You have identified this is a $0 \cdot \infty$ indeterminate form and then transformed it to a 0/0 indeterminate form so you can use L'Hopital's Rule.
	C. Incorrect! Don't forget we have ln(sin *x*) not just sin *x*. This would become cos *x* when differentiating and then when L'Hopital's Rule is applied you would obtain the limit of sin *x* which leads to 1 once the limit is taken.
	D. Incorrect! Don't forget to take the limit after applying L'Hopital's Rule.
	E. Incorrect! You would get this answer if you applied L'Hopital's Rule to the product tan *x* · ln (sin *x*). Remember L'Hopital's Rule only applies to 0/0 or ∞/∞ indeterminate forms. You must change to one of these forms before applying L'Hopital's Rule.
Solution	To evaluate this limit we first identify the limit form. We note that as $x \to \frac{\pi}{2}$, tan *x* → ∞ and sin *x* → 1 so ln(sin *x*) → 0 so we identify this as a $0 \cdot \infty$ indeterminate form. Consequently, we first transform this to a 0/0 or ∞/∞ form so we can apply L'Hopital's Rule. Hence, using tan *x* = 1/cot *x* the limit becomes $\lim_{x \to \pi/2} \left[\tan x \cdot \ln(\sin x)\right] = \lim_{x \to \pi/2} \left[\frac{\ln(\sin x)}{\cot x}\right]$ Now as $x \to \frac{\pi}{2}$, cot *x* → 0 and ln(sin *x*) → 0 so this is now a 0/0 indeterminate form. Applying L'Hopital's Rule produces $\lim_{x \to \pi/2} \left[\frac{\ln(\sin x)}{\cot x}\right] = \lim_{x \to \pi/2} \frac{\frac{d}{dx}(\ln(\sin x))}{\frac{d}{dx}(\cot x)} = \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x}$ Simplifying using the fundamental trigonometric functions yields $= \lim_{x \to \pi/2} \left[-\frac{\cos x}{\sin x} \cdot \frac{\sin^2 x}{1}\right] = \lim_{x \to \pi/2} \left[-\cos x \sin x\right]$ Now take the limit. $\lim_{x \to \pi/2} \left[-\cos x \sin x\right] = -\cos(\pi/2)\sin(\pi/2) = -(0)(1) = 0$ **Solution**: $\lim_{x \to \pi/2} \left[\tan x \cdot \ln(\sin x)\right] = 0$

<table>
<tr><td colspan="2">Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.</td></tr>
<tr><td>Question #03</td><td>3. Evaluate: $\lim_{x \to 0^+} \left[\frac{1+3x}{\sin x} - \frac{1}{x} \right]$

(A) 0
(B) $\frac{7}{2}$
(C) −3
(D) 3
(E) ∞</td></tr>
<tr><td rowspan="5">Feedback on Each Answer Choice</td><td>A. Incorrect!
∞ – ∞ ≠ 0. Also, be careful when checking the limit after using L'Hopital's Rule that you obtain 0/0 and not just 0 in the numerator.</td></tr>
<tr><td>B. Incorrect!
Be sure when you use L'Hopital's Rule for the final time that you differentiate 1as 0 not 1.</td></tr>
<tr><td>C. Incorrect!
Be careful the derivative of sin x is cos x not −cos x.</td></tr>
<tr><td>D. Correct!
Outstanding! You have successfully used the algebraic process of finding a common denominator to change this limit to a 0/0 indeterminate form allowing L'Hopital's Rule to be applied.</td></tr>
<tr><td>E. Incorrect!
Remember as $x \to 0^+$ then $1/x \to \infty$ not 0 and we don't have ∞ – 0 = ∞.</td></tr>
<tr><td>Solution</td><td>We first note that $(1+3x)/\sin x \to \infty$ and $1/x \to \infty$ as $x \to 0^+$. Hence, we have an $\infty - \infty$ indeterminate form. Using the common denominator of $x\sin x$ we transform this limit to
$$\lim_{x \to 0^+} \left[\frac{1+3x}{\sin x} - \frac{1}{x} \right] = \lim_{x \to 0^+} \left[\frac{x(1+3x) - \sin x}{x \sin x} \right].$$
Simplifying produces
$$= \lim_{x \to 0^+} \left[\frac{x + 3x^2 - \sin x}{x \sin x} \right]$$
We now note that as $x \to 0^+$, $3x^2 \to 0^+$, and $\sin x \to 0^+$ which means we now have a 0/0 indeterminate form. Applying L'Hopital's Rule produces
$$\lim_{x \to 0^+} \left[\frac{x + 3x^2 - \sin x}{x \sin x} \right] = \lim_{x \to 0^+} \left[\frac{1 + 6x - \cos x}{1 \cdot \sin x + x \cdot \cos x} \right]$$
$$= \lim_{x \to 0^+} \left[\frac{1 + 6x - \cos x}{\sin x + x \cos x} \right]$$
Again, we notice that as $x \to 0^+$, $1 + 6x - \cos x \to 0^+$ and $\sin x + x\cos x \to 0^+$ so we still have a 0/0 indeterminate form. Consequently, we apply L'Hopital's Rule again to produce
$$= \lim_{x \to 0^+} \left[\frac{6 + \sin x}{\cos x + 1 \cdot \cos x + x \cdot (-\sin x)} \right]$$
$$= \lim_{x \to 0^+} \left[\frac{6 + \sin x}{2\cos x - x \sin x} \right]$$
Taking the limit yields
$$= \left[\frac{6+0}{2(1) - (0)(0)} \right] = \frac{6}{2} = 3.$$
Solution
$$\lim_{x \to 0^+} \left[\frac{1+3x}{\sin x} - \frac{1}{x} \right] = 3$$</td></tr>
</table>

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Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #04	**4.** Evaluate: $$\lim_{x \to 0^+} x^x$$ (A) 0 *(B) e* (C) ∞ (D) −1 (E) 1
Feedback on Each Answer Choice	A. Incorrect! Be sure you don't missing taking the final step of determining that actual limit and not just the limit of the logarithm of the function.
	B. Incorrect! The limit of $(1 + x)^{1/x}$ as $x \to 0$ is equal to e so this cannot be true.
	C. Incorrect! Be careful to simplify the final quotient correctly to $-x$ and not $-1/x$.
	D. Incorrect! Be careful that you differentiate $1/x$ accurately as $-1/x^2$ not $1/x^2$ or the sign will be incorrect.
	E. Correct! Well done! You have mastered the logarithmic requirements to produce a 0/0 indeterminate limit form to which L'Hopital's Rule can be applied.

Solution

We first note that as $x \to 0^+$ we clearly have an 0^0 indeterminate form. We use logarithms to transform this limit. To do this we assume the limit exists and is equal to y. That is,

$$y = \lim_{x \to 0^+} x^x$$

We now take the natural logarithm of both sides. That is,

$$\ln y = \ln\left[\lim_{x \to 0^+} x^x\right]$$

Using the algebra of limits and continuity allows this to be simplified to

$$\ln y = \lim_{x \to 0^+}\left[\ln x^x\right]$$

Using the logarithm property $\ln a^p = p\ln a$ this becomes

$$\ln y = \lim_{x \to 0^+}\left[x \ln x\right]$$

Writing this as a quotient yields

$$\ln y = \lim_{x \to 0^+}\left[\frac{\ln x}{1/x}\right]$$

Consequently, the limit is now a 0/0 indeterminate form. We apply L'Hopital's Rule to produce

$$\ln y = \lim_{x \to 0^+}\left[\frac{\ln x}{1/x}\right] = \lim_{x \to 0^+}\left[\frac{1/x}{-1/x^2}\right] = \lim_{x \to 0^+}\left[-x\right]$$

Taking the limit produces

$$\ln y = \lim_{x \to 0^+}\left[-x\right] = 0$$

Taking e of both sides to find y leads to

$$e^{\ln y} = y = e^0 = 1$$

That is,

$$y = \lim_{x \to 0^+} x^x = 1$$

Solution

$$\lim_{x \to 0^+} x^x = 1$$

Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #05	**5.** Find: $\lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^{3x}$ (A) e^{-6} (B) e^{6} (C) -6 (D) 1 (E) ∞
Feedback on Each Answer Choice	A. Correct! Nice job! You have correctly determined the indeterminate form as 1^{∞} and accurately applied logarithmic techniques.
	B. Incorrect! Be careful to correctly interpret the find result of taking the limit and the exponent laws. Remember, $e^{-6} = 1/e^{6}$.
	C. Incorrect! Make sure you complete the final step of finding the limit required and not the logarithm of the limit.
	D. Incorrect! This is an indeterminate form since if you attempt to take the limit directly the result is 1^{∞}, so indeterminate techniques must be employed since the limit can not determined directly.
	E. Incorrect! Be careful not to misinterpret the limit. This is an indeterminate form since if you attempt to take the limit directly the result is 1^{∞}, so indeterminate techniques must be employed since the limit can not determined directly.

Solution

We first note that as $x \to \infty$, $3x \to \infty$ but $2/x \to 0$ and we clearly have an 1^{∞} indeterminate form. We use logarithms to transform this limit. To do this we assume the limit exists and is equal to y. That is,

$$y = \lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^{3x}$$

We now take the natural logarithm of both sides. That is,

$$\ln y = \ln\left[\lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^{3x}\right]$$

Using the algebra of limits and continuity allows this to be simplified to

$$\ln y = \lim_{x \to \infty} \left[\ln\left(1 - \frac{2}{x}\right)^{3x}\right]$$

Using the logarithm property $\ln a^{p} = p \ln a$ this becomes

$$\ln y = \lim_{x \to \infty} \left[3x \ln\left(1 - \frac{2}{x}\right)\right]$$

Pulling the constant 3 out and writing this as a quotient yields

$$\ln y = 3 \cdot \lim_{x \to \infty} \left[\frac{\ln(1 - 2/x)}{1/x}\right]$$

Consequently, the limit is now a 0/0 indeterminate form. We apply L'Hopital's Rule to produce

$$\ln y = 3 \cdot \lim_{x \to \infty} \left[\frac{\ln(1 - 2/x)}{1/x}\right] = 3 \cdot \lim_{x \to 0^{+}} \left[\frac{[1/(1 - 2/x)][2/x^{2}]}{-1/x^{2}}\right]$$

$$= 3 \cdot \lim_{x \to \infty} \left[\frac{\frac{2/x^{2}}{(1 - 2/x)}}{-1/x^{2}}\right] = -3 \cdot \lim_{x \to \infty} \left[\frac{2}{x^{2}(1 - 2/x)} \cdot \frac{x^{2}}{1}\right]$$

$$= -6 \cdot \lim_{x \to \infty} \left[\frac{1}{(1 - 2/x)}\right]$$

Taking the limit produces

$$\ln y = -6 \cdot \lim_{x \to \infty} \left[\frac{1}{(1 - 2/x)}\right] = (-6)(1) = -6$$

Taking e of both sides to find y leads to

$$e^{\ln y} = y = e^{-6}$$

Solution

$$\lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^{3x} = \frac{1}{e^{6}}$$

Question No. 6 of 10

Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #06	**6.** Evaluate the improper integral: $\int_1^{\infty} \frac{\ln x}{x^2}\, dx$ (A) $-\infty$ (B) ∞ (C) 0 (D) 1 (E) -1
Feedback on Each Answer Choice	A. Incorrect! Check your integration. Also, a negative would suggest that the area under the curve $f(x) = (\ln x)/x^2$ is infinitely negative!
	B. Incorrect! You should notice that the function $(\ln x)/x^2$ is continuous on $[1, \infty)$ which is an infinite or unbounded interval. Consequently, this is an improper integral of Type 1 – Case 1. Check your integration.
	C. Incorrect! This answer is suggesting that there is no area under the curve $f(x) = (\ln x)/x^2$. A sketch of the curve should reveal this is not true. The question really is "Is this area infinite or finite?"
	D. Correct! Outstanding! You have correctly identified this as a Type 1 – Case 1 improper integral problem.
	E. Incorrect! Be careful to carry the negative sign through the integration process. Also, a negative would suggest that the area under the curve $f(x) = (\ln x)/x^2$ is negative!
Solution	To evaluate this integral (if it can be), we first identify the type of improper integral problem. Since the integral is to be determined over the infinite (unbounded) interval $[1, \infty)$ and the integrand $(\ln x)/x^2$ is continuous on the interval $[1, \infty)$ we identify this as Type 1 – Case 1 improper integral. If the improper integral converges the limit will exist and will be a finite number. Since this integral is a Type 1 – Case 1 improper integral we have $$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{c \to \infty} \int_1^{c} \frac{\ln x}{x^2}\, dx$$ We next evaluate the indefinite integral (Omitting the constant of integration). $$\int \frac{\ln x}{x^2}\, dx = \int \left[\ln x \cdot \frac{1}{x^2}\right] dx$$ We integrate this integral using integration by parts with $u = \ln x$, $du = \frac{1}{x}dx$, $dv = \frac{1}{x^2}dx$, and $v = -\frac{1}{x}$. $$= \int \left[\ln x \cdot \frac{1}{x^2}\right] dx = (\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$ $$= -\frac{\ln x}{x} + \int \left(x^{-2}\right) dx = -\frac{\ln x}{x} + \frac{x^{-1}}{-1}$$ $$= -\frac{\ln x}{x} - \frac{1}{x}$$ Hence $$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{c \to \infty} \int_1^{c} \frac{\ln x}{x^2}\, dx = \lim_{c \to \infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^c$$ We evaluate this using the Fundamental Theorem of Calculus $$= \lim_{c \to \infty} \left[\left(-\frac{\ln c}{c} - \frac{1}{c}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{(1)}\right)\right]$$ $$= \lim_{c \to \infty} \left[\left(-\frac{\ln c}{c} - \frac{1}{c}\right) + 1\right]$$ Taking the limit produces $$= \lim_{c \to \infty} \left[\left(-\frac{\ln c}{c} - \frac{1}{c}\right) + 1\right] = \lim_{c \to \infty} \left(-\frac{\ln c}{c}\right) - 0 + 1$$ The remaining limit is an ∞/∞ indeterminate form so we apply L'Hopital's Rule $$= \lim_{c \to \infty} \left(-\frac{\ln c}{c}\right) - 0 + 1 = \lim_{c \to \infty} \left(-\frac{1/c}{1}\right) + 1 = 0 + 1 = 1$$ The integral is convergent. **Solution** $$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = 1$$

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Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #07	**7.** Evaluate (if possible): $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx$ (A) $\pi/2$ (B) $-\pi/2$ (C) $\pi/4$ (D) 0 (E) ∞
Feedback on Each Answer Choice	A. Correct! Nice job, you have identified the type of improper integral problem as a Type 1 – Case 3 improper integral since the integrand $(e^x)/(1 + e^{2x})$ is continuous on the interval (−∞, ∞).
	B. Incorrect! Be careful of your signs particularly when integrating and then recombining the two integrals.
	C. Incorrect! Remember the original integral by definition consists of two improper integrals. The values of these two integrals must be added to determine the value of the original integral.
	D. Incorrect! This suggests that you think that the e^{2x} dominates but the functions are defined on the infinite interval (−∞, ∞) so it cannot be determined what the value is without using improper integral methods. This is a Type 1 – Case 3 improper integral.
	E. Incorrect! This suggests that one of the improper integrals is divergent. You need to check your integration and the values of those integrals when applying the Fundamental Theorem of Calculus. <u>Both</u> integrals are convergent and hence the original integral is also convergent.

Solution

To evaluate this integral (if it can be), we first identify the type of improper integral problem. Since the integral is to be determined over the infinite (unbounded) interval (−∞, ∞) and the integrand $(e^x)/(1 + e^{2x})$ is continuous on the interval (−∞, ∞) we identify this as Type 1 – Case 3 improper integral. If the improper integral converges the limit will exist for both the improper integrals that form this integral and will be a finite number. Since this integral is a Type 1 – Case 3 improper integral we have using the convenient choice $a = 0$,

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx = \lim_{c \to -\infty} \int_c^0 \frac{e^x}{1+e^{2x}}\, dx + \lim_{c \to \infty} \int_0^c \frac{e^x}{1+e^{2x}}\, dx$$

We need to first integrate the indefinite integral (Omitting the constant of integration.)

$$\int \frac{e^x}{1+e^{2x}} dx = \int \frac{e^x}{1+\left(e^x\right)^2} dx$$

We integrate by substitution and letting $u = e^x$, so $du = e^x\, dx$. The integral now becomes

$$\int \frac{e^x}{1+e^{2x}} dx = \int \frac{e^x}{1+\left(e^x\right)^2} dx = \int \frac{du}{1+u^2}$$

Integrating by using $u = \tan\theta$ with $du = \sec^2\theta\, d\theta$, and the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$ yields

$$\int \frac{du}{1+u^2} = \int \frac{\sec^2\theta d\theta}{1+\tan^2\theta} = \int \frac{\sec^2\theta d\theta}{\sec^2\theta} = \int d\theta = \theta$$

We now need to reintroduce u using $u = \tan\theta$ or $\theta = \tan^{-1}(u)$. But $u = e^x$ allows the reintroduction of x. Consequently,

$$\int \frac{e^x}{1+e^{2x}} dx = \int \frac{du}{1+u^2} = \tan^{-1} u = \tan^{-1}\left(e^x\right)$$

Now evaluate each of the integrals making up original integral separately

$$\lim_{c \to -\infty} \int_c^0 \frac{e^x}{1+e^{2x}}\, dx = \lim_{c \to -\infty} \left[\tan^{-1}(e^x)\right]_c^0 = \lim_{c \to -\infty} \left[\tan^{-1}(e^0) - \tan^{-1}(e^c)\right] = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

and

$$\lim_{b \to \infty} \int_0^b \frac{e^x}{1+e^{2x}}\, dx = \lim_{b \to \infty} \left[\tan^{-1}(e^x)\right]_0^b = \lim_{b \to \infty} \left[\tan^{-1}(e^b) - \tan^{-1}(e^0)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Putting the two integrals together shows that the original integral is convergent and

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx = \lim_{c \to -\infty} \int_c^0 \frac{e^x}{1+e^{2x}}\, dx + \lim_{c \to \infty} \int_0^c \frac{e^x}{1+e^{2x}} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Solution

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx = \frac{\pi}{2}$$

Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #08	**8.** Determine the convergence conditions in terms of p for the following integral. $$\int_0^1 \frac{1}{x^p}\,dx$$ (A) The integral is convergent for all values of p. (B) The integral is divergent for all values of p. (C) The integral is convergent for $p \geq 1$ and divergent for $p < 1$. (D) The integral is divergent for $p \geq 1$ and convergent for $p < 1$. (E) The integral is divergent for $p > 1$ and convergent for $p < 1$.
Feedback on Each Answer Choice	A. Incorrect! Be careful to take into consideration the effect on the integral of the exponent p when $p < 1$, $p = 1$, and $p > 1$. Consider the $p = 1$ case first and you will see that the corresponding integral diverges.
	B. Incorrect! Be careful to take into consideration of the effect of the exponent p when $p < 1$, $p = 1$, and $p > 1$. Consider the simple case when $p = 2$ case and you will see that the corresponding integral converges.
	C. Incorrect! Be careful that you interpret the resulting integral cases correctly in terms of convergence and divergence.
	D. Correct! Outstanding! You have recognized this as a Type 2 – Case 2 improper integral problem. Further, you have discovered that important effect on the integral of the exponent p when $p < 1$, $p = 1$, and $p > 1$.
	E. Incorrect! Be careful not to forget the important but simple case of when $p = 1$.
Solution	To determine convergence for this integral we first must determine the type of improper integral problem that this represents. Since the integral is to be determined over the interval (0, 1], and the integrand $1/x^p$ is continuous on the interval (0, 1], we identify this as Type 2 – Case 2 improper integral. In terms of p we can consider three different scenarios that affect the integral due to p being the exponent on x. These scenarios are when $p = 1$, $p < 1$, and $p > 1$. We will consider each of these cases separately. First, suppose $p = 1$, the integral becomes $$\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{x^1}\,dx = \int_0^1 \frac{1}{x}\,dx = \lim_{c \to 0^+} \int_c^1 \frac{1}{x}\,dx$$ Integrating produces $$= \lim_{c \to 0^+} \left[\ln\lvert x\rvert\right]_c^1 = \lim_{c \to 0^+} \left[\ln 1 - \ln c\right] = \lim_{c \to 0^+} \left[-\ln c\right] = \infty$$ Consequently, we see that the integral diverges when $p = 1$. Now suppose the $p \neq 1$, $$\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 x^{-p}\,dx = \lim_{c \to 0^+} \int_c^1 x^{-p}\,dx$$ Integrating produces $$\lim_{c \to 0^+} \int_c^1 x^{-p}\,dx = \lim_{c \to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_c^1 = \lim_{c \to 0^+} \left[\frac{1^{-p+1}}{-p+1} - \frac{c^{-p+1}}{-p+1}\right]$$ $$= \lim_{c \to 0^+} \left[\frac{1}{1-p} - \frac{1}{1-p} \cdot \frac{1}{c^{p-1}}\right]$$ If $p < 1$, then when the limit is taken, $1/c^{p-1} \to 0$, and we have convergence. Consequently, $$\lim_{c \to 0^+} \int_c^1 x^{-p}\,dx = \frac{1}{1-p}$$ If $p > 1$, then when the limit is taken, $1/c^{p-1} \to \infty$, and we have divergence. Consequently, $$\lim_{c \to 0^+} \int_c^1 x^{-p}\,dx = \infty$$ **Solution** $$\lim_{c \to 0^+} \int_c^1 x^{-p}\,dx = \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases}$$

Question No. 9 of 10

Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #09	**9.** Evaluate: $\int_0^2 \frac{dx}{\sqrt[3]{x-1}}$ (A) 3/2 (B) 0 (C) −3/2 (D) ∞ (E) 1
Feedback on Each Answer Choice	A. Incorrect! Two improper integrals make up this improper integral. Make sure you add the results of each of these integrals to determine the final result. This answer is the answer to one of the improper integrals.
	B. Correct! Well done, you correctly identified this as a Type 3 – Case 3 improper integral. An improper integral can converge to 0.
	C. Incorrect! Two improper integrals make up this improper integral. Make sure you add the results of each of these integrals to determine the final result. This answer is the answer to one of the improper integrals.
	D. Incorrect! This suggests that one of the integrals that this integral consists of is divergent. You need to check your integration and evaluations when using the Fundamental Theorem of Calculus.
	E. Incorrect! Check your integration that you have integrated correctly an not omitted the constant that comes from dividing by the exponent.

Solution

First, we observe that the integrand in the interval [0, 2] has a discontinuity at $x = 1$. Therefore, we identify this as a Type 3 – Case 3 improper integral problem. Consequently, $c = 1$ and the integral becomes

$$\int_0^2 \frac{dx}{\sqrt[3]{x-1}} = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^2 \frac{dx}{\sqrt[3]{x-1}}$$

or

$$\int_0^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt[3]{x-1}} + \lim_{r \to 1^+} \int_1^2 \frac{dx}{\sqrt[3]{x-1}}$$

We now consider the following indefinite integral (Omitting the constant of integration).
Letting $u = x - 1$ and $du = dx$ we have

$$\int \frac{dx}{\sqrt[3]{x-1}} = \int \frac{du}{\sqrt[3]{u}} = \int \left(u^{-\frac{1}{3}}\right) du$$

Integrating and replacing u with $u = x - 1$ produces

$$\int \left(u^{-\frac{1}{3}}\right) du = \frac{u^{\frac{2}{3}}}{2/3} = \frac{3}{2}(x-1)^{\frac{2}{3}} = \frac{3}{2}\sqrt[3]{(x-1)^2}$$

We now evaluate each of the improper integrals using the Fundamental Theorem of Calculus, that is

$$\lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt[3]{x-1}} = \lim_{t \to 1^-} \left[\frac{3}{2} \cdot \sqrt[3]{(x-1)^2}\right]_0^t = \frac{3}{2} \cdot \lim_{t \to 1^-} \left[\sqrt[3]{(t-1)^2} - \sqrt[3]{(0-1)^2}\right] = \frac{3}{2} \cdot \left[\sqrt[3]{(0)^2} - \sqrt[3]{(-1)^2}\right] = \frac{3}{2} \cdot [0-1] = -\frac{3}{2}$$

and

$$\lim_{r \to 1^+} \int_r^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{r \to 1^+} \left[\frac{3}{2} \cdot \sqrt[3]{(x-1)^2}\right]_r^2 = \frac{3}{2} \cdot \lim_{r \to 1^+} \left[\sqrt[3]{(2-1)^2} - \sqrt[3]{(r-1)^2}\right] = \frac{3}{2} \cdot \left[\sqrt[3]{(1)^2} - \sqrt[3]{(0)^2}\right] = \frac{3}{2} \cdot [1-0] = \frac{3}{2}$$

We now recombine the integrals. Consequently,

$$\int_0^2 \frac{dx}{\sqrt[3]{x-1}} = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt[3]{x-1}} + \lim_{r \to 1^+} \int_1^2 \frac{dx}{\sqrt[3]{x-1}} = -\frac{3}{2} + \frac{3}{2} = 0$$

Hence, since both limits exist (both integrals converge) the improper integral converges.

Solution

$$\int_0^2 \frac{dx}{\sqrt[3]{x-1}} = 0$$

Question No. 10 of 10

Instructions: (1) Read the problem and answer choices carefully (2) Work the problems on paper as needed (3) Pick the answer (4) Go back to review the core concept tutorial as needed.	
Question #10	**10.** By performing a suitable convergence test what can be said about the convergence or divergence of: $$\int_0^{\pi/2} \frac{dx}{\sqrt{3x}+\sin 2x}$$ (A) Converges for all real x. (B) Diverges for all real x. (C) Converges for all x in the interval $[0, \pi/2]$. (D) Diverges for all x in the interval $[0, \pi/2]$. (E) Converges when x is in the interval $[0, \pi/4]$ and diverges when x is in the interval $[\pi/4, \pi/2]$.
Feedback on Each Answer Choice	A. Incorrect! Be careful to identify the interval that you are applying the convergence test to as it cannot be applied unless all the conditions of the test are satisfied. Identify these conditions first.
	B. Incorrect! Be careful to identify the interval that you are applying the convergence test to as it cannot be applied unless all the conditions are satisfied. Also, check the convergence result for the integral you are using as a comparison.
	C. Correct! Well done, you identified the Direct Comparison Test to use and the integral of the function you need to use as a comparison.
	D. Incorrect! Check the convergence result for the integral you are using as a comparison.
	E. Incorrect! Identify the convergence tests you should use as the Direct Comparison Test and the correct function that needs to be integrated for the comparison. This should be $g(x)=1/\sqrt{3x}$.
Solution	To test for convergence we can use the Direct Convergence Test which states that if two functions f and g are continuous on a closed interval $[a, b]$ and $0 \le f(x) \le g(x)$ for all x in $[a, b]$, then $$\int_a^b f(x)\,dx \text{ converges if } \int_a^b g(x)\,dx \text{ converges}$$ For this case $a = 0$ and $b = \frac{\pi}{2}$. Also since, $\sin 2x \ge 0$ when $0 \le x \le \frac{\pi}{2}$ then we choose $$g(x)=\frac{1}{\sqrt{3x}} \text{ and } f(x)=\frac{1}{\sqrt{3x}+\sin 2x}$$ and see that $0 \le f(x) \le g(x)$ when $0 \le x \le \frac{\pi}{2}$. Consequently, the Direct Comparison Test can be applied. We first check the convergence the Type 2 – Case 1 improper integral $$\int_0^{\pi/2}\frac{dx}{\sqrt{3x}} = \lim_{c\to 0^+}\int_c^{\pi/2}(3x)^{-1/2}dx = \lim_{c\to 0^+}\int_c^{\pi/2}(3)^{-1/2}(x)^{-1/2}dx = \frac{1}{\sqrt{3}}\cdot\lim_{c\to 0^+}\int_c^{\pi/2}\left(x^{-1/2}\right)dx$$ Integrating using the Fundamental Theorem of Calculus produces $$\frac{1}{\sqrt{3}}\cdot\lim_{c\to 0^+}\int_c^{\pi/2}\left(x^{-1/2}\right)dx = \frac{1}{\sqrt{3}}\cdot\lim_{c\to 0^+}\left[\frac{x^{1/2}}{1/2}\right]_c^{\pi/2} = \frac{2}{\sqrt{3}}\cdot\lim_{c\to 0^+}\left[\sqrt{x}\right]_0^{\pi/2}$$ Taking the limit yields $$=\frac{2}{\sqrt{3}}\cdot\lim_{c\to 0^+}\left[\sqrt{\frac{\pi}{2}}-\sqrt{c}\right]=\sqrt{\frac{2\pi}{3}}$$ We conclude that $$\int_0^{\pi/2}\frac{dx}{\sqrt{3x}} \text{ converges}$$ So by the Direct Comparison Test since $\int_0^{\pi/2}\frac{dx}{\sqrt{3x}}$ converges then $\int_0^{\pi/2}\frac{dx}{\sqrt{3x}+\sin 2x}$ converges **Solution** $$\int_0^{\pi/2}\frac{dx}{\sqrt{3x}+\sin 2x} \text{ converges}$$

