02: Limits and Continuity

Key Terms

- **The Limit of $f(x)$ as x Approaches a**: The value to which the output (dependent variable) of $f(x)$ approaches as x (independent variable) approaches the number a.
- **Polynomial**: A mathematical expression consisting of one or more summed terms, each term consisting of a constant multiplier and one or more variables raised to integral powers.
- **Continuous**: A function is continuous at a particular x-value if the function is defined at that value, and if the limit as the function approaches that value exists and equals the output of the function at the value.
- **Discontinuity**: A function has a discontinuity at a particular x-value if it is undefined at that value, or if it is not continuous at that value.
- **Vertical Asymptote**: A vertical line that the graph of a function gets unboundedly close to but never reaches. The function is undefined at the x-value where the vertical asymptote occurs.
- **Removable Discontinuity**: A single point that the graph of a function "skips." Usually the result of having a common factor in the numerator and denominator of the function, which can be crossed out.
- **∞**: "Infinity" is not a number, but the idea of something greater than any positive real number. Similarly, $-\infty$ is something less than any negative real number.

Approximating Limits Numerically

1. Identify the independent variable and the value it is approaching.
   $\lim_{t \to 4}(1-t^2)$ In the above limit, the independent variable is t and it is approaching 4.
2. Pick two numbers, one slightly smaller and one slightly larger than the value the independent variable is approaching (within 0.01 often works; so try 3.99 and 4.01) and plug those numbers into the function.
3. If the results are relatively close to one another, pick a number in between the two as your approximation for the value of the limit.
4. If the results seem far apart, repeat steps 2 and 3 with numbers closer to 4, like 3.999 and 4.001. If the results are still far apart, the limit might not exist.

Finding Limits Algebraically

Plug the number the independent variable is approaching into the function and simplify. The results you get will determine what method you should use to find the limit. If you get…

- …a finite number, then that number is the limit.

Plugging in was all you had to do!

- … $\frac{a}{0}$, where a is a finite number, then there is a **vertical asymptote** at the value x is approaching.
- … $\frac{0}{0}$, then there is a **removable discontinuity** at the value x is approaching.

Limits at Vertical Asymptotes

If you determine there is a vertical asymptote at the value x is approaching, follow these steps to find the limit:

- Pick numbers a little less and a little more than the value x is approaching (such as within 0.01)
- Plug these numbers into the function. You should get large positive or negative values.
- If both values are large positive numbers, the limit is ∞.
- If both values are large negative numbers, the limit is $-\infty$.
- If one value is positive and one value is negative, the limit does not exist.

Limits at Removable Discontinuities

If you determine $f(x)$ has a removable discontinuity at the value x is approaching, follow these steps to find the limit:

- Factor the numerator and denominator of the function.
- Look for a common factor in the numerator and denominator, and cancel it to simplify the function.

Plug the value x is approaching back into the simplified function. The limit equals whatever number you get!

Limits – Problem Solving Tips

- A graph of the function will tell you a lot about the function. If you have access to one, or can sketch one yourself, you can use it to approximate the limit before applying a numerical or algegraic method.
- Identify the independent variable (often x, but not always) and the value it is approaching (a). This is found below the abbreviation "Lim" in the form $x \to a$.
- Plug the value the independent variable is approaching into the function to determine the method you should use (plugging in, vertical asymptote or removable discontinuity)
- If you can factor and simplify, do so!
- You can often find the limit of polynomials and many other simple functions by evaluating $f(x)$ at $x = a$.

Some Common Discontinuities

If there's no reason for a function to have a discontinuity, it is likely continuous. Below are some of the more common discontinuities.

- Negative values inside even numbered roots.
- Vertical asymptotes
- Removable discontinuities
- Piecewise discontinuities.

Some Functions That are Continuous Everywhere

Functions that are continuous everywhere can be relied upon to behave predictably, so it's good to be familiar with them. In particular, finding limits of continuous functions is very easy, because you can just plug the x-value being approached into the function.
Here are some of the most common functions that are continuous everywhere:

- Polynomials
- Exponential Functions
- Some trigonometric functions (sinx and cosx, and basic combinations of them)
- Rational functions with no real zeroes in the denominator

Continuity Problem Solving Tips

- Try to recognize the function as one that is continuous everywhere, or one that has discontinuities.
- If the function has discontinuities, what kind are they? Are they undefined values, vertical asymptotes, or removable discontinuities?
- To determine whether a piecewise function is continuous everywhere, first decide whether the individual equations are continuous over their domains. Then plug each transition value into the equations surrounding them. If the two functions on either side have equal values, then the piecewise function is continuous there too.