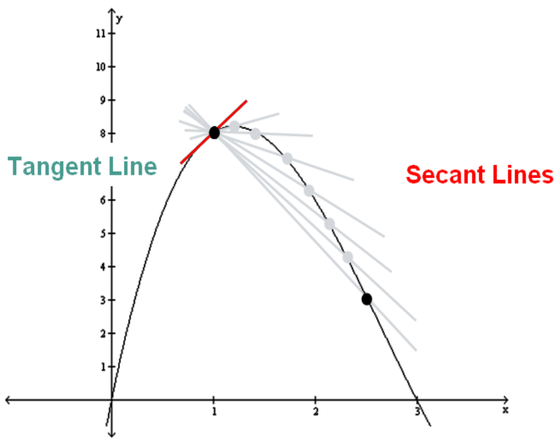
10: The Derivative and the Tangent Line

Key Terms

- **Secant Line:** A secant line is a line that passes through two points that lie on a curve.
- **Tangent Line:** A tangent line to a curve at a given point is a line that barely touches the curve at that point.
- **Average Rate of Change**: The average rate of change is the average rate at which a function changes over a given interval. It is represented by the slope of a secant line.
- **Instantaneous Rate of Change**: The instantaneous rate of change is the rate of change of the function at a single point. It is represented by the slope a tangent line.
- **Derivative:** The derivative of a function $f(x)$ is another function $f'(x)$ which gives the slope of the tangent line for any value of x.

Tangent Line

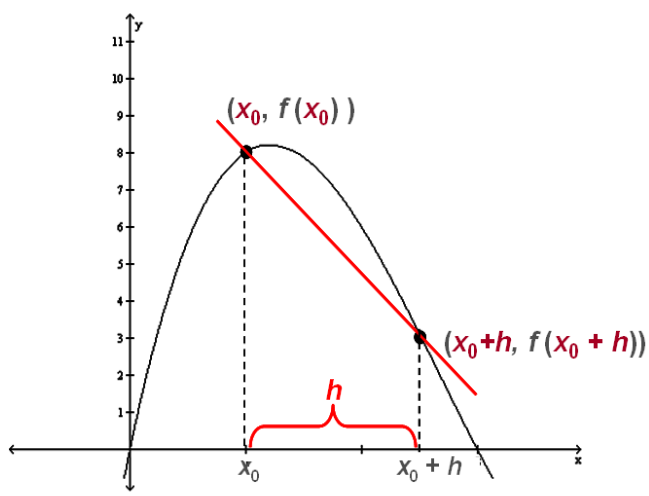
The secant lines converge to the tangent line as the two points get closer and closer.



The Difference Quotient

The difference quotient is defined to the be slope of the secant line:

$$Slope = \frac{f(x_0 + h) - f(x_0)}{h}$$



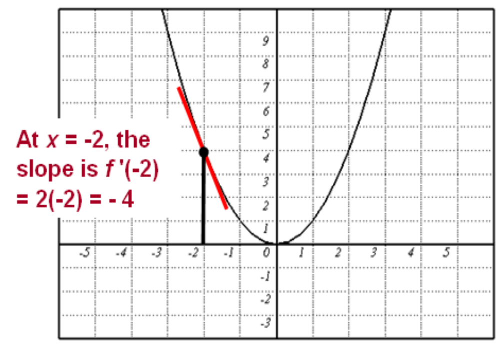
Definition of the Derivative

The derivative, denoted by $f'(x)$ gives the slope of the tangent line to a curve $f(x)$. Since the secant line approaches the tangent line when $h \to 0$, the slope of the secant line (i.e., the difference quotient) approaches the slope of the tangent line (i.e., the derivative) when $h \to 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Interpretation of the Derivative

Suppose that $f(x) = x^2$. The derivative is $f'(x) = 2x$. Now consider the point (-2, 4) on the graph of $f(x)$. The slope of the tangent line at the point (-2, 4) would be given by $f'(-2) = 2(-2) = -4$.



Example

Example: Find the derivative of $f(x) = -2x^2$.

Solution:
The derivative is:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{-2(x+h)^2 - (-2x^2)}{h} \\
&= \lim_{h \to 0} \frac{-2(x^2 + 2xh + h^2) + 2x^2}{h} \\
&= \lim_{h \to 0} \frac{-2x^2 - 4xh - 2h^2 + 2x^2}{h} \\
&= \lim_{h \to 0} \frac{\cancel{-2x^2} - 4xh - 2h^2 \cancel{+2x^2}}{h} \\
&= \lim_{h \to 0} \frac{-4xh - 2h^2}{h} \\
&= \lim_{h \to 0} \frac{h(-4x - 2h)}{h} \\
&= \lim_{h \to 0} (-4x - 2h) \\
&= -4x
\end{aligned}$$

Horizontal Tangent Line

A tangent line is horizontal whenever the derivative equals zero.

Continuity

If the derivative of a function exists at a point, the function is guaranteed to be continuous at that point.

How to Use This Cheat Sheet: These are the keys related this topic. Try to read through it carefully twice then rewrite it out on a blank sheet of paper. Review it again before the exams.